HOMOLOGY FIBRATIONS AND "GROUP-COMPLETION" REVISITED

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Abstract. We give a proof of the Jardine-Tillmann generalized group completion theorem. It is much in the spirit of the original homology fibration approach by McDuff and Segal, but follows a modern treatment of homotopy colimits, using as little simplicial technology as possible. We compare simplicial and topological definitions of homology fibrations.

Introduction

The group completion of a topological monoid M is the loop space ΩBM and a group completion theorem is originally a statement about the relation between the homology of M and that of ΩBM. In the appendix of [8] D. Quillen considers a simplicial monoid M. His main theorem is that under certain conditions the homology of the group completion of M can be computed by inverting $\pi_0 M$ in the homology of M. A similar result can be found in May's [13, Theorem 15.1]. In this paper we focus on a more topological kind of group completion theorem, the question being how to construct ΩBM out of M. Our starting point is McDuff's and Segal's theorem, as it can be found in [15, Proposition 2] (a good account on the subject is Adams' book on infinite loop spaces [1, Chapter 3]).

Theorem *Let M be a topological monoid acting on a space X by homology equivalences. Then the map $\pi : EM \times_M X \to BM$ from the Borel construction to the classifying space of M is a homology fibration with fibre X.*

The standard application is as follows. Let M be a homotopy commutative topological monoid with $\pi_0 M \cong \mathbb{N}$. Choose a point m in the component of 1 and form the telescope $M_\infty = Tel(M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} \dots)$. The action of M by left multiplication on M_∞ is by homology equivalences because M is homotopy commutative. Hence we obtain:

Corollary *Let M be a homotopy commutative topological monoid. Then there is a homology equivalence $M_\infty \to \Omega BM$. Moreover, when $\pi_1 M_\infty$ is perfect, $\Omega BM \simeq M_\infty^+$.*

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Taking for example M to be the disjoint union $\coprod B\Sigma_n$ of classifying spaces of the symmetric groups, the Barrat-Priddy-Quillen Theorem states that $B\Sigma_\infty^+$ is the infinite loop space QS^0, [4]. Likewise, taking M to be $\coprod BGL_n(R)$ one gets back Quillen's definition of the algebraic K-theory of a ring R, [16].

Simplicial versions of the group completion theorem started appearing at the end of the eighties. I. Moerdijk provides a homological statement in [14, Corollary 3.1] and J.F. Jardine the analogue of the above theorem in [11, Theorem 4.2], which he calls the "strong form of the Group Completion Theorem". More recently U. Tillmann introduced a "multiple object case" in her celebrated work on the stable mapping class group ([18, Theorem 3.2]). In this context the Borel construction is replaced by a bisimplicial version, i.e. the realization of a certain simplicial space. Let $\mathcal{M}$ be a simplicial category and $F : \mathcal{M}^{op} \to Spaces$ a contravariant diagram. There is always a natural transformation to the trivial diagram. Taking the bisimplicial Borel constructions yields a map $\pi_{\mathcal{M}} : E_{\mathcal{M}}F \to B\mathcal{M}$, analogous to the map π in the classical theorem.

Theorem 3.2. *Let $\mathcal{M}$ be a simplicial category and $F : \mathcal{M}^{op} \to Spaces$ a contravariant diagram. Assume that any morphism $f : i \to j$ induces an isomorphism in integral homology $H_*(F(j); \mathbb{Z}) \to H_*(F(i); \mathbb{Z})$. Then, for each object $i \in \mathcal{M}$, the map $F(i) \to Fib_i(\pi_{\mathcal{M}})$ to the homotopy fibre of $\pi_{\mathcal{M}}$ over i is a homology equivalence.*

We offer in this paper a proof which uses as little simplicial technology as possible. The main ingredient is a rather classical result about comparing the fibre of the realization with the realization of the fibres, an idea already used by McDuff and Segal in their proof of the classical group completion theorem. Of course we do not avoid simplicial spaces, the theorem after all is about delooping a simplicial classifying space. We work however more in the spirit of the modern theory homotopy colimits. One very powerful tool in this setting is to decompose a space as a diagram over its simplices. The advantage of this approach is that one gets a more geometric feeling about the constructions performed (such as the bisimplicial Borel construction). We also use a simplicial notion of homology fibrations (preimages of simplices have the same integral homology as the homotopy fibre). In the last section we compare this concept to that of classical homology fibration in the category of topological spaces and prove they coincide.

In this paper space means simplicial set and we write *Spaces* for the category of spaces.

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1. Homology fibrations

Let $p : E \to B$ be a map of spaces and σ be an n-simplex in B. We denote by $dp(\sigma)$ the pull-back of the diagram $\Delta[n] \xrightarrow{\sigma} B \xleftarrow{p} E$. This is the "preimage" of the simplex in E and yields a functor $dp : \Delta B \to \mathit{Spaces}$ from the simplex category of the base space (this category is defined for example in [6, p.182], see also [5, Definition 6.1]). It allows to decompose the map p as a diagram over ΔB, as one has $E \simeq \mathit{hocolim}_{\Delta B} dp$ and $B \simeq \mathit{hocolim}_{\Delta B} \Delta[n]$.

We will also need a slight generalization of dp, replacing a simplex by any space K. For a map $f : K \to B$, define $dp(f)$ to be the pull-back of f along p.

Definition 1.1. A map of spaces $p : E \to B$ is a *homology fibration* if the natural map $dp(\sigma) \to Fib_\sigma(p)$ to the homotopy fibre of p over the component of σ is a homology equivalence for any simplex $\sigma \in B$. It is a *weak homology fibration* if for any simplex $\sigma \in B$ and any simplicial operation θ we have a homology equivalence $dp(\sigma) \to dp(\theta\sigma)$.

The aim of this section is to prove that a weak homology fibration is actually a homology fibration. This part of the paper replaces Segal and McDuff's work on locally contractible paracompact spaces.

Lemma 1.2. [15, Proposition 6] *Let $p : E \to B$ be a weak homology fibration with B contractible. Then p is a homology fibration.*

Proof. The category ΔB is contractible since $B \simeq \mathit{hocolim}_{\Delta B} * = N(\Delta B)$. So E is equivalent to the homotopy colimit over a contractible category of a diagram in which all maps are homology equivalences. This homotopy colimit has the same homology type as any of the values $dp(\sigma)$ since it can be computed ([2]) by using only push-outs and telescopes of diagrams consisting of homology equivalences. We conclude by the Mayer–Vietoris Theorem and the fact that homology commutes with telescopes. □

Proposition 1.3. *Let $p : E \to B$ be a weak homology fibration and $f : B' \twoheadrightarrow B$ a fibration. The pull-back of p along f is another weak homology fibration $p' : E' \to B'$.*

Proof. Let σ' be a simplex in B', $\sigma = f\sigma'$ its image in B and θ any simplicial operation. Then $dp(\sigma)$ has the same homology type as $dp(\theta\sigma)$ by assumption. But $dp'(\sigma') \simeq dp(\sigma)$ and $dp'(\theta\sigma') \simeq dp(\theta\sigma)$ since p' was obtained as a pull-back. □

Theorem 1.4. [15, Proposition 5] *A weak homology fibration is a homology fibration.*

Proof. Let $p : E \to B$ be a weak homology fibration and choose $f : PB \twoheadrightarrow B$ the path space fibration. The above proposition applies, so $p' : Fib_\sigma(p) \to PB$ is a weak homology fibration as well for any simplex σ in B. Since f is surjective, there exists a simplex $\sigma' \in PB$ such that $f(\sigma') = \sigma$. Therefore $dp(\sigma) \simeq dp'(\sigma')$, which has the same homology type as the homotopy fibre $Fib_\sigma(p)$ by Lemma 1.2. □

2. Realizations and fibres

Theorem 1.4 will be used throughout this section. For checking that a map is a homology fibration it suffices to check it is a weak homology fibration.

Lemma 2.1. *Consider a commutative square*

$$\begin{tikzcd} E_0 \arrow[r] \arrow[d, "p_0"'] & E_1 \arrow[d, "p_1"] \\ B_0 \arrow[r] & B_1 \end{tikzcd}$$

where the vertical arrows are compatible homology fibrations in the sense that the map $Fib_v(p_0) \to Fib_v(p_1)$ is an integral homology equivalence for any vertex $v \in B_0$. Then $dp_0(f) \to dp_1(f)$ is an integral homology equivalence for any map $f : K \to B_0$. Moreover if both horizontal maps are cofibrations, then so is $dp_0(f) \to dp_1(f)$.

Proof. Notice first that if σ is a simplex in B_0, then $dp_0(\sigma) \to dp_1(\sigma)$ is an integral homology equivalence by our assumption on the homotopy fibres over vertices. Likewise the preimages in E_0 and E_1 of a disjoint union of simplices have the same integral homology type. We assume therefore that K is connected. Assume $K = L \cup_{\partial\Delta[n]} \Delta[n]$. By induction on the dimension suppose that both $dp_0(f|_L) \to dp_1(f|_L)$ and $dp_0(f|_{\partial\Delta[n]}) \to dp_1(f|_{\partial\Delta[n]})$ are homology equivalences. We see that the preimage of $\partial\Delta[n]$ is contained in that of $\Delta[n]$ so that

$$dp_0(f) = colim\big(dp_0(f|_L) \leftarrow dp_0(f|_{\partial\Delta[n]}) \hookrightarrow dp_0(f|_{\Delta[n]})\big)$$

is actually a homotopy push-out. Thus $dp_0(f) \to dp_1(f)$ is a homotopy push-out of homology equivalences. □

We prove now that a push-out of homology fibrations is still a homology fibration. As everybody knows a map can always be replaced by a fibration, so we must pay close attention to the constructions we perform. We always use strict colimits, but for diagrams where the colimit is weakly equivalent to the homotopy colimit.

Proposition 2.2. *Consider a natural transformation between push-out diagrams:*

$$\begin{array}{ccccccccc}
E & = & colim & \big(& E_1 & \longleftarrow & E_0 & \hookrightarrow & E_2 & \big) \\
{\scriptstyle p}\downarrow & & & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle p_2} & \\
B & = & colim & \big(& B_1 & \longleftarrow & B_0 & \hookrightarrow & B_2 & \big)
\end{array}$$

such that $p_n : E_n \to B_n$ is a homology fibration for $0 \leq n \leq 2$ and the right hand-side horizontal maps are cofibrations. Assume that the map $Fib_v(p_0) \to Fib_v(p_n)$ is an integral homology equivalence for any vertex $v \in B_0$ if $n = 1, 2$. Then p is a homology fibration as well. Moreover, if for some $0 \leq n \leq 2$, w is a vertex in B_n, then $B_n \hookrightarrow B$ induces a homology equivalence $Fib_w(p_n) \to Fib_w(p)$.

Proof. Any simplex σ in B lies either in B_1 or in B_2. Say it lies in B_1 (the other case is similar) and consider the pull-back K of $\Delta[n] \to B_1 \leftarrow B_0$. Apply Lemma 2.1 to the map $f : K \to B_0$ to conclude that $dp_0(f) \to dp_2(f)$ is a homology equivalence, which is even a cofibration. Hence the preimage $dp(\sigma)$ is the (homotopy) push-out $colim\big(dp_1(\sigma) \leftarrow dp_0(f) \hookrightarrow dp_2(f)\big)$. The homotopy push-out of a homology equivalence is again a homology equivalence so that $dp(\sigma)$ has the same homology type as $dp_1(\sigma)$. We conclude that p is a weak homology fibration. $\square$

Proposition 2.3. *Consider a natural transformation between telescope diagrams:*

$$\begin{array}{cccccccccccc}
E & = & colim & \big(& E_0 & \hookrightarrow & E_1 & \hookrightarrow & E_2 & \hookrightarrow & \cdots & \big) \\
{\scriptstyle f}\downarrow & & & & {\scriptstyle p_0}\downarrow & & {\scriptstyle p_1}\downarrow & & {\scriptstyle p_2}\downarrow & & & \\
B & = & colim & \big(& B_0 & \hookrightarrow & B_1 & \hookrightarrow & B_2 & \hookrightarrow & \cdots & \big)
\end{array}$$

such that $p_n : E_n \to B_n$ is a homology fibration for any $n \geq 0$ and all horizontal maps are cofibrations. Assume that the map $Fib_v(p_n) \to Fib_v(p_{n+1})$ is an integral homology equivalence for any $n \geq 0$ and any vertex $v \in B_n$. Then p is a homology fibration as well. Moreover, if w is a vertex in B_n for some $n \geq 0$, then the inclusion $B_n \hookrightarrow B$ induces a homology equivalence $Fib_w(p) \to Fib_w(p_n)$.

Proof. As $B = \bigcup B_n$, any simplex σ of B lies in some B_N. The conclusion follows since $dp(\sigma) = \bigcup_{n \geq N} dp_n(\sigma)$ has the same homology type as $dp_N(\sigma)$. $\square$

Let $X_\bullet$ be a simplicial space. Recall that Segal's thick realization $||X_\bullet||$ ([17, Appendix A]) is defined by an inductive process. We have $||X_\bullet|| = \bigcup_n ||X_\bullet||_n$ where $||X_\bullet||_0 = X_0$ and $||X_\bullet||_n$ is constructed from $||X_\bullet||_{n-1}$ by the following push-out

$$colim\big(||X_\bullet||_{n-1} \leftarrow \partial\Delta[n] \times X_n \hookrightarrow \Delta[n] \times X_n\big)$$

and the map $\partial\Delta[n] \times X_n \to ||X_\bullet||_{n-1}$ is defined using only the face maps. This thick realization can be seen as the homotopy colimit of the diagram $X_\bullet$ over the subcategory of Δ^{op} generated by the face morphisms.

Theorem 2.4. [15, Proposition 4] *Let $p_\bullet : E_\bullet \to B_\bullet$ be a map of simplicial spaces such that $p_n : E_n \to B_n$ is a weak homology fibration for any $n \geq 0$. Assume that any face map $d_i : [n] \to [n+1]$ induces an integral homology equivalence on homotopy fibres $Fib_v(p_{n+1}) \to Fib_{d_i v}(p_n)$ for any vertex $v \in B_{n+1}$. Then $p : ||E_\bullet|| \to ||B_\bullet||$ is a homology fibration as well. Moreover, if w is a vertex in $||B_\bullet||$ lying in the same connected component as a vertex $v \in B_n$, then there is a homology equivalence $Fib_w(p) \to Fib_v(p_n)$.*

Proof. Each step is a homotopy push-out involving only the face maps, so Proposition 2.2 applies. Hence $||p_\bullet||_n$ is a homology fibration for any $n \geq 0$ and we conclude by Proposition 2.3. □

One could actually prove a more general statement involving a colimit over a small indexing category instead of the realization of a simplicial space. In this paper we will not need such a statement.

3. The generalized group completion

The aim is to find a model for the loops on the classifying space of a simplicial category. Let us start with a brief reminder on simplicial categories. More details can be found for example in [18, Section 1], especially about the link with 2-categories. Roughly speaking a simplicial category is a category equipped with spaces of morphisms instead of sets of morphisms. So $mor_{\mathcal{M}}(i,j)$ is a space for any objects $i, j \in \mathcal{M}$ and $mor_{\mathcal{M}}(i,i)$ contains the identity morphism as distinguished base point. More precisely a simplicial category $\mathcal{M}$ is a simplicial object in the category of small categories with constant object set. It is helpful to look at $\mathcal{M}$ as a functor $\Delta^{op} \to CAT$, where the category of n-simplices is the category having same objects as $\mathcal{M}$ and morphisms from i to j are the n-simplices of the space of morphisms from i to j. Taking now the nerve of this simplicial category degree by degree produces a simplicial space denoted by $B\mathcal{M}_\bullet$, the *simplicial classifying space*.

A contravariant diagram $F : \mathcal{M}^{op} \to Spaces$ is the data of spaces $F(i)$ for all objects $i \in \mathcal{M}$ and natural continuous maps $\mu_{i,j} : mor_{\mathcal{M}}(i,j) \times F(j) \to F(i)$. The simplicial category itself produces an example of diagram with $\mathcal{M}(i) = \coprod_{j \in Obj(\mathcal{M})} mor_{\mathcal{M}}(i,j)$.

Definition 3.1. The *bisimplicial Borel construction* of a diagram $F : \mathcal{M}^{op} \to Spaces$ is the simplicial space $E_{\mathcal{M}}F_\bullet$ whose space of n-simplices is the disjoint union over all

n-tuples of objects in $\mathcal{M}$

$$\coprod_{i_0,\dots,i_n} mor_{\mathcal{M}}(i_n, i_{n-1}) \times \cdots \times mor_{\mathcal{M}}(i_1, i_0) \times F(i_0)$$

The degeneracy maps are the obvious inclusions. The face map $d_n : E_{\mathcal{M}}F_n \to E_{\mathcal{M}}F_{n-1}$ is projection on the last n factors, $d_0 = 1 \times \mu_{i_1,i_0}$, and the other d_k's are defined by composition $mor_{\mathcal{M}}(i_{k+1}, i_k) \times mor_{\mathcal{M}}(i_k, i_{k-1}) \to mor_{\mathcal{M}}(i_{k+1}, i_{k-1})$.

The trivial diagram $T(i) = \{i\}$ is the diagram in which any morphism $i \to j$ induces the unique map $\{j\} \to \{i\}$. The bisimplicial Borel construction of the trivial diagram is nothing but the simplicial classifying space of $\mathcal{M}$, i.e. $E_{\mathcal{M}}T_\bullet = B\mathcal{M}_\bullet$. Every diagram $F : \mathcal{M}^{op} \to Spaces$ comes with a natural transformation $\pi : F \to T$ and hence we get a map of simplicial spaces

$$E_{\mathcal{M}}\pi_\bullet : E_{\mathcal{M}}F_\bullet \to B\mathcal{M}_\bullet.$$

The preimage of $\{i\}$ in the bisimplicial Borel construction is $F(i)$. Denote by $E_{\mathcal{M}}F$ the realization $||E_{\mathcal{M}}F_\bullet||$, by $B\mathcal{M}$ the realization $||B\mathcal{M}_\bullet||$, and by $\pi_{\mathcal{M}} : E_{\mathcal{M}}F \to B\mathcal{M}$ the map induced by π. We are ready to prove now the main theorem.

Theorem 3.2. [18, Theorem 3.2] *Let $\mathcal{M}$ be a simplicial category and $F : \mathcal{M}^{op} \to Spaces$ a contravariant diagram. Assume that any morphism $f : i \to j$ induces an isomorphism in integral homology $H_*(F(j); \mathbb{Z}) \to H_*(F(i); \mathbb{Z})$. Then, for each object $i \in \mathcal{M}$, the map $F(i) \to Fib_i(\pi_{\mathcal{M}})$ to the homotopy fibre of $\pi_{\mathcal{M}}$ over i is a homology equivalence.*

Proof. We apply Theorem 2.4 to the map $E_{\mathcal{M}}\pi_\bullet$. For any $n \geq 0$, the map $E_{\mathcal{M}}F_n \to B\mathcal{M}_n$ is the projection on the first factors, thus a (homology) fibration. As all faces but d_0 induce the identity on the fibres, we have only to check that the face map d_0 induces a homology equivalence on the fibres. Choose a vertex

$$(f_n, \dots, f_1, i_0) \in mor_{\mathcal{M}}(i_n, i_{n-1}) \times \cdots \times mor_{\mathcal{M}}(i_1, i_0) \times \{i_0\}$$

Its zeroth face is $(f_n, \dots, f_2, i_1)$ and the map induced on the homotopy fibres is $F(f_0) : F(i_0) \to F(i_1)$. This is a homology equivalence by assumption and we are done. □

In order to identify the space $\Omega B\mathcal{M}$ we need to find a diagram F which satisfies the assumptions of Theorem 3.2 and for which the bisimplicial Borel construction $E_{\mathcal{M}}F$ is contractible. We give a partial answer to that question which covers the applications made in the context of the mapping class group.

Let us consider for any object $j \in \mathcal{M}$ the diagram $\mathcal{M}_j$ as defined in [18, Section 3]. It is the restriction of the diagram $\mathcal{M}$, i.e. $\mathcal{M}_j(i) = mor_{\mathcal{M}}(i, j)$. This diagram has a

contractible bisimplicial Borel construction $E_{\mathcal{M}}\mathcal{M}_j \simeq *$ (see [18, Lemma 3.3]. Now fix an object $1 \in \mathcal{M}$ and an endomorphism $\alpha : 1 \to 1$, i.e. a vertex in the space of morphisms $mor_{\mathcal{M}}(1,1)$. Form the diagram $\mathcal{M}_\infty(i) = hocolim(\mathcal{M}_1(i) \xrightarrow{\alpha_*} \mathcal{M}_1(i) \xrightarrow{\alpha_*} \ldots)$. Since homotopy colimits commute with themselves $E_{\mathcal{M}}\mathcal{M}_\infty \simeq hocolim E_{\mathcal{M}}\mathcal{M}_1$ is contractible and the homotopy fibre of $\pi_{\mathcal{M}}$ is $\Omega B\mathcal{M}$. We apply now the theorem to the diagram $\mathcal{M}_\infty$.

Proposition 3.3. *Let $\mathcal{M}$ be a simplicial category and assume that there exists an endomorphism α of a specific object* 1 *such that any morphism $f : i \to j$ induces an integral homology equivalence $\mathcal{M}_\infty(j) \to \mathcal{M}_\infty(i)$. Then the natural map $\mathcal{M}_\infty(i) \to \Omega B\mathcal{M}$ is an integral homology equivalence for any object $i \in \mathcal{M}$.* □

Finally one particularly likes the case when $\Omega B\mathcal{M}$ can be identified as Quillen's plus construction applied to the space $\mathcal{M}_\infty(1)$. This means that the map $\mathcal{M}_\infty(1) \to \Omega B\mathcal{M}$ is not only a homology equivalence, but an acyclic map (its homotopy fibre is acyclic). When is this so? In general a homology equivalence is acyclic if the fundamental group of the base space acts nilpotently on the homology of the homotopy fibre (assuming the fibre is connected, see [3, 4.3 (xii)]). This is usually rather difficult to verify. A stronger condition is that $\pi_1\mathcal{M}_\infty(1)$ is perfect. Then indeed every component of $\mathcal{M}_\infty(1)^+$ is 1-connected and hence $\mathcal{M}_\infty(1)^+$ is an $H\mathbb{Z}$-local space. Consider now the following commutative square in which all arrows are homology equivalences

$$\begin{tikzcd}
\mathcal{M}_\infty(1) \arrow[r] \arrow[d] & \Omega B\mathcal{M} \arrow[d] \\
\mathcal{M}_\infty(1)^+ \arrow[r] & (\Omega B\mathcal{M})^+
\end{tikzcd}$$

First $(\Omega B\mathcal{M})^+ \simeq \Omega B\mathcal{M}$ since the fundamental group of any component of a loop space is abelian. Moreover a loop space is always $H\mathbb{Z}$-local, so that $\mathcal{M}_\infty(1)^+ \to \Omega B\mathcal{M}$ is a homology equivalence between $H\mathbb{Z}$-local spaces, thus a homotopy equivalence.

The above condition on the diagram $\mathcal{M}_\infty$ are precisely those checked in the proof of [18, Theorem 3.1] to identify the plus construction on the classifying space of the stable mapping class group as a loop space, which turns then out to be an infinite loop space.

Remark 3.4. The homology theory which has been used in the present work is integral homology and all applications we know of are obtained working with integral homology. However, with little effort one can replace this homology theory by an arbitrary (possibly extraordinary) homology theory E_*. Hence an E_**-fibration* is a map $p : E \to B$ such that $dp(\sigma) \to Fib_\sigma(p)$ is an E_*-equivalence. This is equivalent to require that p be a *weak*

E_**-fibration*, i.e. $dp(\sigma) \to dp(\theta\sigma)$ is an E_*-equivalence for any simplex σ in B and any simplicial operation θ. Then one can prove the analogous of Theorem 2.4: The realization of a natural transformation $p_\bullet : E_\bullet \to B_\bullet$ of simplicial spaces where all fibers have the same E_*-homology is an E_*-fibration. The generalized group completion theorem has an E_*-analogue as well, and the question would then be to compare the homotopy type of $\Omega B\mathcal{M}$ with the E_*-theoretic plus construction.

4. Simplices versus topology

The general idea behind simplicial sets is to replace topological data (points) by a combinatorial one (simplices). This is precisely why one defines simplicially a homology fibration by imposing a condition on the preimages of simplices, instead of classically looking at preimages of points. There is however a subtle difference, as shown by the following example due to W. Waldhausen, which we learned from J. Rognes during the BCAT02. A *simple map* of topological spaces is a map $f : X \to Y$ such that the preimages of points $f^{-1}(y) \simeq *$ are contractible for all $y \in Y$. Thus one would be tempted to define simplicially a simple map as a map of spaces $f : X \to Y$ for which preimages of simplices $dp(\sigma) \simeq *$ are all contractible. This is not equivalent to the topological definition. Consider indeed your favorite (but non-trivial) acyclic space A. The map $A \to *$ induces one on the unreduced suspensions $\Sigma A \to \Delta[1]$. The preimage of the simplices in $\Delta[1]$ are either points, or ΣA, so all are contractible. But topologically the geometric realization of this map is not simple because the preimage of any other point than the end points of the interval is A.

Recall that a map of topological spaces is a homology fibration if the preimages of all points have the same homology type as the homotopy fibre of p. We prove in this section that the simplicial and topological definitions of homology fibrations are equivalent. Basically this is due to the Mayer–Vietoris Theorem. The idea is to take the barycentric subdivision of the map and reconstruct the preimage of the barycenter of a simplex in the base from the data given by the preimages of the simplices. Let us first recall some standard definitions from [12] (or [9, Chapter 4]).

Let μ be a proper face of $\Delta[n]$. We denote by k_μ the dimension of μ, that is μ is an injection $\mu\ : \Delta[k_\mu] \hookrightarrow \Delta[n]$. The *barycentric subdivision* of $\Delta[n]$, denoted by $\Delta'[n]$, is the space which has as q-simplices μ the increasing sequences of $q+1$ faces of $\Delta[n]$, i.e. $\mu = (\mu_0, \cdots, \mu_q)$ where $\mu_i(\Delta[k_i]) \subset \mu_{i+1}(\Delta[k_{i+1}])$ for all $i \leq q-1$. The simplicial operations are the usual: If $\theta\ : \Delta[q] \longrightarrow \Delta[p]$ is any simplicial operation then $\Delta'\alpha(\mu) = (\mu_{\theta(0)}, \cdots, \mu_{\theta(q)})$.

The *subdivision* functor Sd is left adjoint to Kan's extension functor Ex (see [12, Section 7]). For any space E, the q-simplices of SdE are by definition the equivalence classes $[x, \mu]$ of a simplex $x \in E$ of dimension p and $\mu \in \Delta'[p]$ of dimension q. Two pairs (x, μ) and (x', μ') are equivalent if there exists a map $\alpha : \Delta[p'] \to \Delta[p]$ such that $x' = x\alpha$ and $\mu = \Delta'\alpha(\mu)$. In other words, SdE is the colimit over the simplex category of E of the subdivisions of these simplices: $SdE = colim_{\Delta E}\Delta[n]'$.

Let us fix a surjective map $f : E \to \Delta[n]$. Its subdivision $Sdf : SdE \to \Delta'[n]$ is defined as follows. Let $[x, \mu]$ be a simplex in SdE as above and consider for any $0 \le i \le q$ the composite

$$\Delta[k_i] \xrightarrow{\mu_i} \Delta[p] \xrightarrow{x} E \xrightarrow{f} \Delta[n]$$

It can be decomposed in a unique way as a degeneracy followed by an injection $\Delta[k_i] \xrightarrow{\phi_i} \Delta[l_i] \overset{\nu_i}{\hookrightarrow} \Delta[n]$. Set $f([x, \mu]) = \nu = (\nu_0, \ldots, \nu_q)$.

Definition 4.1. In $\Delta'[n]$ fix a vertex α, i.e. a proper face of $\Delta[n]$. The *star* of α, $St(\alpha)$ is the subspace of $\Delta'[n]$ which has as simplices the sequences $(\mu_0, \cdots, \mu_p)$ such that $\forall i \le p$, $\mathrm{Im}\mu_i \supset \mathrm{Im}\alpha$. We will further denote by $ESt(\alpha)$ the preimage of $St(\alpha)$ under Sdf.

Lemma 4.2. *The inclusion $Sdf^{-1}(\alpha) \hookrightarrow ESt(\alpha)$ is a homotopy equivalence.*

Proof. Let α be of dimension k. We construct first a retraction $r : ESt(\alpha) \to Sdf^{-1}(\alpha)$. Let $[x, \mu] \in ESt(\alpha)$ be a simplex of dimension q. Then, for any $i \le q$, there exists a maximal injective morphism $\Delta[t_i] \hookrightarrow \Delta[k_i]$ (determined by the vertices of μ_i whose image under $f(x)$ is a vertex of α) together with a (necessary unique) surjection $\phi : \Delta[t_i] \to \Delta[k]$ rendering the following diagram commutative

$$\begin{tikzcd}
\Delta[k_i] \arrow[r, hook, "\mu_i"] & \Delta[p] \arrow[r, "x"] & E \arrow[d, "f"] \\
\Delta[t_i] \arrow[u, hook] \arrow[r, "\phi"] & \Delta[k] \arrow[r, "\alpha"] & \Delta[n]
\end{tikzcd}$$

We denote the composite $\Delta[t_i] \to \Delta[k_i] \to \Delta[p]$ by $\bar{\mu}_i$ and define $r[x, \mu] = [x, \bar{\mu}]$. By construction $Sdf([x, \bar{\mu}])$ is some degeneracy of α. Moreover r is well defined and is clearly a retraction of the inclusion $i : Sdf^{-1}(\alpha) \hookrightarrow ESt(\alpha)$.

Finally we construct a homotopy $H : ESt(\alpha) \times \Delta[1] \to ESt(\alpha)$ from $i \circ r$ to the identity. Let $([x, \mu], \tau)$ be a q-simplex in the cylinder, so τ is a q-simplex in $\Delta[1]$ and can be represented by a sequence of $r + 1$ zero's and $q - r$ one's: $(0 \ldots 01 \cdots 1)$. Define then $H([x, \mu], \tau) = [x, \bar{\mu}_0, \ldots, \bar{\mu}_r, \mu_{r+1}, \ldots, \mu_q]$. ∎

In the next proposition we use the decomposition of $\Delta'[n]$ as union of all its stars. More precisely consider the category $\mathcal{C}_n$ whose objects are the non-degenerate simplices of $\Delta[n]$ and whose morphisms are generated by the faces $\sigma \to d_i\sigma$. The unique non-degenerate simplex τ of dimension n is an initial object and diagrams indexed by $\mathcal{C}_n$ are n-cubes without terminal object. We have $\Delta'[n] = colim_{\sigma\in\mathcal{C}_n} St(\sigma) = hocolim_{\sigma\in\mathcal{C}_n} St(\sigma)$ because the diagram St is cofibrant (see for example [7]), and even strongly co-Cartesian as defined in [10, Definition 2.1]. Likewise

$$E \simeq SdE = colim_{\sigma\in\mathcal{C}_n} ESt(\sigma) = hocolim_{\sigma\in\mathcal{C}_n} ESt(\sigma)$$

Proposition 4.3. *Let $f : E \to \Delta[n]$ be a homology fibration. Then the preimage of the barycenter of $\Delta'[n]$ under Sdf has the same homology type as E. In particular the realization $|f| : |E| \to |\Delta[n]|$ is a homology fibration of topological spaces.*

Proof. By Lemma 4.2 the values of the cubical diagram ESt are equivalent to the preimages $Sdf^{-1}(\sigma)$. When σ is a vertex of $\Delta[n]$, one has that $Sdf^{-1}(\sigma) \simeq f^{-1}(\sigma) = df(\sigma)$, which by hypothesis has the same homology type as E. By induction on the dimension of σ we can assume thus that all values in the diagram but the initial one ($ESt(\tau) \simeq Sdf^{-1}(\tau)$, the preimage of the barycenter) are homology equivalent to E. As the homotopy colimit of the cubical diagram is E, we deduce that $ESt(\tau)$ as well has the same homology type as E. We claim that this implies that $|f|$ is a (topological) homology fibration. Indeed by induction again we need only to compute preimages under $|f|$ of points in the interior of the realization of $\Delta[n]$. Any such preimage is a deformation retract of the preimage under $|p|$ of the open simplex, so it is enough to consider the barycenter. The above computation shows precisely that it has the same homology type as $|E|$, the homotopy fibre of $|f|$. □

Let us now consider a map $p : E \to B$. To compare both types of homology fibrations we need to control the homological properties of fibers of points in the realization of spaces. Any point $b \in |B|$ lies in the interior of the realization of a unique non-degenerate simplex $\sigma_b \in B$ (see for instance [9, Lemma 4.2.5]). Moreover the interior of the realization of σ_b embeds in $|B|$.

Theorem 4.4. *A map of spaces $p : E \to B$ is a homology fibration if and only if its realization $|p| : |E| \to |B|$ is a homology fibration of topological spaces.*

Proof. First assume that $p : E \to B$ is a homology fibration. We need to compute the homology type of fibers of points in the realization of B and show that the map

$|p|^{-1}(b) \to Fib_b(|p|)$ is a homology equivalence , where $Fib_b(|p|)$ denotes the homotopy fiber of $|p|$ over the connected component of b. When $\sigma = \sigma_b$ is a 0-simplex, this is trivial as p is a homology fibration. If σ is of dimension $n \geq 1$, notice that all the fibers over the points in the interior of $|\sigma|$ have the same homotopy type (a straightforward computation shows then that the preimage any point is a deformation retract of the preimage under $|p|$ of the open simplex). Therefore it suffices to analyze the barycenter ι_n of the realization of σ and to prove that $|p|^{-1}(\iota_n) \to Fib_{\iota_n}(|p|)$ is a homology equivalence. As the realization functor commutes with finite limits (see [9, Theorem 4.3.16]), we have a pull-back square :

$$\begin{tikzcd}
{|dp(\sigma)|} \arrow[r] \arrow[d] & {|E|} \arrow[d] \\
{|\Delta[n]|} \arrow[r, "\sigma"] & {|B|}
\end{tikzcd}$$

The map $dp(\sigma) \to \Delta[n]$ is a homology fibration as the pull-back of any simplex of the base $\Delta[n]$ coincides with the pull-back of a simplex of B along p, which has the same homology type as $dp(\sigma)$. By Proposition 4.3, its realization is a homology fibration: The preimage of the barycenter of $|\Delta[n]|$ is homology equivalent to the homotopy fibre $|dp(\sigma)|$, which by assumption has the same homology type as the homotopy fibre $|F|$ of $|p|$.

Assume now $|p| : |E| \to |B|$ is a homology equivalence. Inductively we may suppose that for all simplices of dimension $\leq n-1$ the pull-back $dp(\tau)$ is homology equivalent to the homotopy fibre above the component of τ. Let σ be a simplex of dimension n. We have as before a pull-back diagram

$$\begin{tikzcd}
{|dp(\sigma)|} \arrow[r] \arrow[d] & {|E|} \arrow[d] \\
{|\Delta[n]|} \arrow[r, "\tau"] & {|B|}
\end{tikzcd}$$

Decompose $dp(\sigma)$ as a cubical homotopy colimit $dp(\sigma) \simeq hocolim_{\tau \in \mathcal{C}_n} ESt(\tau)$ following the method seen in the proof of Proposition 4.3. As $|p|$ is a homology fibration, there is a natural transformation by homology equivalences to the constant cubical diagram $Fib_\sigma(p)$ (use Lemma 4.2). A homotopy colimit of homology equivalences is a homology equivalence, hence $dp(\sigma) \to Fib_\sigma(p)$ is a homology equivalence as well. □

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